Carleman estimate for the Navier-Stokes equations and an application to a lateral Cauchy problem

Mourad Bellassoued

Department of Mathematics, Faculty of Sciences of Bizerte

University of Carthage, 7021 Jarzouna Bizerte, Tunisia

e-mail: mourad.bellassoued@fsb.rnu.tn

Oleg Imanuvilov

Department of Mathematics, Colorado State University

101 Weber Building, Fort Collins, CO 80523-1874, USA

e-mail: oleg@math.colostate.edu*

Masahiro Yamamoto

Graduate School of Mathematical Sciences, the University of Tokyo

3-8-1 Komaba, Meguro-ku, Tokyo, 153-8914, Japan.

e-mail: myama@ms.u-tokyo.ac.jp†

Abstract

We consider the nonstationary linearized Navier-Stokes equations in a bounded domain and first we prove a Carleman estimate with a regular weight function. Second we apply the Carleman estimate to a lateral Cauchy problem for the

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Navier-Stokes equations and prove the Hölder stability in determining the velocity and pressure field in an interior domain.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain with smooth boundary $\partial\Omega$ (e.g., of C^2-class), and let $\nu = \nu(x)$ be the outward unit normal vector on $\partial\Omega$ at x. We set $Q := \Omega \times (0, T)$.

We consider the linearized Navier-Stokes equations for an incompressible viscous fluid:

$$\partial_t v(x,t) - \kappa\Delta v(x,t) + (A\cdot\nabla)v + (v\cdot\nabla)B + \nabla p = F(x,t) \quad \text{in } Q, \tag{1.1}$$

and

$$\operatorname{div} v(x,t) = 0 \quad \text{in } Q. \tag{1.2}$$

Here $v = (v_1, \cdots, v_n)^T$, $n = 2, 3$, $\cdot^T$ denotes the transpose of matrices, $\kappa > 0$ is a constant describing the viscosity, and for simplicity we assume that the density is one. Let $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$, $1 \le j \le n$, $\Delta = \sum_{j=1}^n \partial_j^2$, $\nabla = (\partial_1, \cdots, \partial_n)^T$, $\nabla_{x,t} = (\nabla, \partial_t)^T$, $\partial_x^\beta = \partial_1^{\beta_1}\cdots\partial_n^{\beta_n}$ with $\beta = (\beta_1, \cdots, \beta_n) \in (\mathbb{N}\cup\{0\})^n$, $|\beta| = \beta_1 + \cdots + \beta_n$,

$$(w\cdot\nabla)v = \left(\sum_{j=1}^n w_j\partial_j v_1, \cdots, \sum_{j=1}^n w_j\partial_j v_n\right)^T$$

for $v = (v_1, \cdots, v_n)^T$ and $w = (w_1, \cdots, w_n)^T$. Throughout this paper, we assume

$$A \in W^{2,\infty}(Q), \quad \nabla B \in L^\infty(Q). \tag{1.3}$$

In this paper, we establish a Carleman estimate with a regular weight function and apply it to a lateral Cauchy problem for the Navier-Stokes equations and prove the Hölder stability in an arbitrarily given interior domain. For stating the main results, we introduce notations. Let I_n be the $n\times n$ identity matrix and let the stress tensor $\sigma(v,p)$ be defined by the $n\times n$ matrix

$$\sigma(v,p) := \kappa(\nabla v + (\nabla v)^T) - pI_n,$$

where κ is some positive constant. We assume

$$d \in C^2(\overline{\Omega}), \quad |\nabla d(x)| > 0 \quad \text{on } \overline{\Omega} \tag{1.4}$$

and we arbitrarily choose $t_0 \in (0,T)$ and $\beta > 0$. We set

$$\psi(x,t) = d(x) - \beta(t-t_0)^2, \quad \varphi(x,t) = e^{\lambda\psi(x,t)}$$

with a sufficiently fixed large constant $\lambda > 0$. We choose a non-empty relatively open subboundary $\Gamma \subset \partial\Omega$ arbitrarily.

Let $D \subset Q$ be a bounded domain with smooth boundary ∂Q such that $\overline{D \cap (\partial\Omega \times (0,T))} \subset \Gamma \times (0,T)$.

For $k, \ell \in \mathbb{N} \cup \{0\}$, we set

$$H^{k,\ell}(D) = \{v \in L^2(D);\, \partial_x^\beta v \in L^2(D),\, |\beta| \le k,\, \partial_t^j v \in L^2(D) \quad 0 \le j \le \ell\}$$

and

$$\|(v,p)\|^2_{\mathcal{X}_s(D)} := \int_D \left\{ \frac{1}{s^2}\left(|\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i\partial_j v|^2 \right) + |\nabla v|^2 + s^2|v|^2 + \frac{1}{s}|\nabla p|^2 + s|p|^2 \right\} e^{2s\varphi} dxdt.$$

We are ready to state our Carleman estimate.

Theorem 1.
There exist constants $s_0 > 0$ and $C > 0$, independent of s, such that

$$\|(v,p)\|^2_{\mathcal{X}_s(D)} \le C\int_D |F|^2 e^{2s\varphi} dxdt + C\int_D (|h|^2 + |\nabla_{x,t}h|^2) e^{2s\varphi} dxdt$$
$$+ Ce^{Cs}(\|v\|^2_{L^2(0,T;H^{\frac{3}{2}}(\Gamma))} + \|\partial_t v\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} + \|\sigma(v,p)\nu\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))}) \tag{1.5}$$

for all $s \ge s_0$ and $(v,p) \in H^{2,1}(D) \times H^{1,0}(D)$ satisfying (1.1),

$$div\, v = h \quad in\ D\ with\ h \in H^{1,1}(D),$$

and

$$\begin{cases} v(\cdot,0) = v(\cdot,T) = 0 & in\ \Omega, \\ |v| = |\nabla v| = |p| = 0 & on\ \partial D \setminus (\Gamma \times (0,T)). \end{cases} \tag{1.6}$$

This is a Carleman estimate for the linearized Navier-Stokes equations (1.1) with (1.2) with boundary data on $\Gamma \subset \partial\Omega$.

Boulakia [2] proves a Carleman estimate with a weight function similar to ours for the homogeneous Stokes equations: $\partial_t v = \Delta v - \nabla p$ and $\mathrm{div}\, v = 0$ with extra interior or boundary data. The Carleman estimate in [2] requires a stronger norm of boundary data than our Carleman estimate if it is applied to the case of the Stokes equations.

As for other Carleman estimates for the Navier-Stokes equations, we refer to Choulli, Imanuvilov, Puel and Yamamoto [3], Fernández-Cara, Guerrero, Imanuvilov and Puel [6], where the authors use a weight function in the form

$$\exp\left(\frac{2sw(x)}{t(T-t)}\right)$$

with some function w and the weight function decays to 0 at $t = 0, T$ exponentially. Their Carleman estimates hold over the whole domain Q for v satisfying $v = 0$ on $\partial\Omega$ but not necessarily $v(\cdot, 0) = v(\cdot, T) = 0$. Those global Carleman estimate is convenient for proving the Lipschitz stability for an inverse source problem (e.g., [3]) and the exact null controllability ([6]), but is not suitable for proving the unique continuation, and such a weight function does not admit Carleman estimates for the Navier-Stokes equations coupled with first-order equation or hyperbolic equation such as a conservation law. As for Carleman estimates for the Navier-Stokes equations, see also Fan, Di Cristo, Jiang and Nakamura [4] and Fan, Jiang and Nakamura [5] with extra data in a neighborhood of the whole boundary, which is too much by considering the parabolicity of the equations.

2 Proof of Theorem 1

First Step.
Let $E \subset \mathbb{R}^n$ be a bounded domain with smooth boundary ∂E and let $E_\delta := \{x \in E;\ \mathrm{dist}\ (x, E) > \delta\}$ with small $\delta > 0$.

We prove
Lemma 1.
Let $p \in H^1(E)$ satisfy

$$\Delta p = f_0 + \sum_{j=1}^{n} \partial_j f_j \quad \text{in } E$$

and $\mathrm{supp}\, p \subset E_\delta$. Let $d_0 \in C^2(\overline{E})$ satisfy $d_0(x) > 0$ for $x \in E$ and $|\nabla d(x)| > 0$ for $x \in E_\delta$. We set $\varphi_0(x) = e^{\lambda d_0(x)}$ with large constant $\lambda > 0$. Then there exist constants $C > 0$ and $s_1 > 0$ such that

$$\int_E \left(\frac{1}{s}|\nabla p|^2 + s|p|^2\right) e^{2s\varphi_0(x)} dx \le C \int_E \left(\frac{1}{s^2}|f_0|^2 + \sum_{j=1}^{n} |f_j|^2\right) e^{2s\varphi_0(x)} dx$$

for all $s \ge s_1$. The constants C and s_1 are independent of choices of p.
Proof. Since ∂E is of C^3-class, we choose a function $\mu \in C^3(\overline{E})$ such that $0 \le \mu \le 1$,

$\mu > 0$ in E and $\mu = \begin{cases} 0, & \text{in } \mathbb{R}^n \setminus E, \\ 1, & \text{in } E_{\delta/2}. \end{cases}$. We set $\widetilde{d_0}(x) = \mu(x)d_0(x)$ and $\widetilde{\varphi_0}(x) = e^{\lambda\widetilde{d_0}(x)}$ for $x \in \overline{E}$. Then $\widetilde{d_0}(x) = 0$ for $x \in \partial E$ and $\widetilde{d_0} > 0$, $|\nabla\widetilde{d_0}| = |\mu\nabla d_0 + d_0\nabla\mu| = |\mu\nabla d_0| > 0$ in E_δ. Hence the H^{-1}-Carleman estimate for an elliptic operator by Imanvilov and Puel [9] yields

$$\int_E \left(\frac{1}{s}|\nabla p|^2 + s|p|^2\right) e^{2s\widetilde{\varphi_0}(x)}dx \le C\int_E \left(\frac{1}{s^2}|f_0|^2 + \sum_{j=1}^n |f_j|^2\right) e^{2s\widetilde{\varphi_0}(x)}dx$$

for all $s \ge s_1$. Here we note that in Theorem 1.2 in [9], we set $\omega = E \setminus \overline{E_\delta}$ and use $p|_\omega = 0$. Since $p = 0$ in $E \setminus E_\delta$ and $\widetilde{d_0} = d_0$ in E_δ, we complete the proof of Lemma 1.

Lemma 2.
There exist constants $s_0 > 0$ and $C > 0$ such that

$$\|(v,p)\|^2_{\mathcal{X}_s(Q)} \le C\int_Q |F|^2 e^{2s\varphi}dxdt + C\int_Q (|h|^2 + |\nabla_{x,t}h|^2)e^{2s\varphi}dxdt \tag{2.1}$$

for all $s \ge s_0$ and $(v,p) \in H^{2,1}(Q) \times H^{1,0}(Q)$ satisfying (1.1),

$$v(\cdot,0) = v(\cdot,T) = 0 \quad \text{in } \Omega,$$

$$|v| = |\nabla v| = |p| = 0 \quad \text{in } \partial\Omega \times (0,T),$$

and

$$div\, v = h \quad in\ Q$$

with some $h \in H^{1,1}(Q)$.

Proof of Lemma 2. Thanks to the large parameter $s > 0$, in view of (1.3), it is sufficient to prove Lemma 1 for $B = 0$ in (1.1). In fact, the Carleman estimate with $B \ne 0$ follows from the case of $B = 0$ by replacing F by $F - (v\cdot\nabla)B$ and estimating $|(F - (v\cdot\nabla)B)(x,t)| \le |F(x,t)| + C|v(x,t)|$ for $(x,t) \in Q$. Then, choosing $s_0 > 0$ large, we can absorb the term $\int_Q |v|^2 e^{2s\varphi}dxdt$ into the left-hand side of the Carleman estimate.

By the density argument, it is sufficient to prove the lemma for (v,p) such that $\mathrm{supp}\, v$ and $\mathrm{supp}\, p$ are compact in Q. We consider

$$\partial_t v = \kappa\Delta v - (A\cdot\nabla)v - \nabla p + F \tag{2.2}$$

and

$$\mathrm{div}\, v = h \qquad \text{in } Q. \tag{2.3}$$

Taking the divergence of (2.2) and using (2.3), we obtain

$$\Delta p = -\sum_{j,k=1}^{n} \{\partial_j((\partial_k A_j)v_k) - (\partial_j \partial_k A_j)v_k\} + \mathrm{div}\, F - \partial_t h - (A\cdot\nabla)h + \kappa \mathrm{div}\,(\nabla h) \quad \text{in } Q. \tag{2.4}$$

Here we used

$$\begin{aligned}\mathrm{div}\,((A\cdot\nabla)v) &= \sum_{j,k=1}^{n} \partial_k(A_j\partial_j v_k) = \sum_{j=1}^{n} A_j\partial_j\left(\sum_{k=1}^{n}\partial_k v_k\right) + \sum_{j,k=1}^{n}(\partial_k A_j)\partial_j v_k \\ &= A\cdot\nabla(\mathrm{div}\, v) + \sum_{j,k=1}^{n}\{\partial_j((\partial_k A_j)v_k) - (\partial_j\partial_k A_j)v_k\}. \end{aligned} \tag{2.5}$$

Moreover on the right-hand side of (2.4), the term $\kappa\mathrm{div}\,(\nabla h)$ is not in $L^2(Q)$ because we assume only $h \in H^{1,1}(Q)$. Thus we cannot apply a usual Carleman estimate requiring $\Delta p \in L^2(Q)$, and we need the H^{-1}-Carleman estimate.

By a usual density argument, we can assume that supp $p \subset Q$. By supp $p \subset Q$, fixing $t \in [0,T]$, we apply Lemma 1 to (2.4) and obtain

$$\begin{aligned}&\int_\Omega \left(\frac{1}{s}|\nabla p(x,t)|^2 + s|p(x,t)|^2\right) e^{2s\varphi(x,t_0)}dx \\ \le\, & C\int_\Omega (|F|^2 + |\partial_t h|^2 + |\nabla h|^2 + |h|^2)e^{2s\varphi(x,t_0)}dx + C\int_\Omega |v(x,t)|^2 e^{2s\varphi(x,t_0)}dx\end{aligned} \tag{2.6}$$

for $s \ge s_1$ where $s_1 > 0$ is a sufficiently large constant.

Let $s_0 := s_1 e^{\lambda\beta T^2}$. Then, $s \ge s_0$ implies

$$se^{-\lambda\beta(t-t_0)^2} \ge se^{-\lambda\beta T^2} \ge s_1$$

for $0 \le t \le T$, so that for fixed $t \in [0,T]$ by replacing s by $se^{-\lambda\beta(t-t_0)^2}$, by (2.5) we can see

$$\begin{aligned}&\int_\Omega \left(\frac{1}{s}|\nabla p(x,t)|^2 + s|p(x,t)|^2\right) \exp(2(se^{-\lambda\beta(t-t_0)^2})\varphi(x,t_0))dx \\ \le & C\int_\Omega (|F|^2 + |\partial_t h|^2 + |\nabla h|^2 + |h|^2)\exp(2(se^{-\lambda\beta(t-t_0)^2})\varphi(x,t_0))dx \\ +&C\int_\Omega |v(x,t)|^2 \exp(2(se^{-\lambda\beta(t-t_0)^2})\varphi(x,t_0))dx,\end{aligned}$$

that is,

$$\int_\Omega \left(\frac{1}{s}|\nabla p(x,t)|^2 + s|p(x,t)|^2\right) e^{2s\varphi(x,t)}dx$$

$$\leq C \int_\Omega (|F|^2 + |\partial_t h|^2 + |\nabla h|^2 + |h|^2) e^{2s\varphi(x,t)} dx + C \int_\Omega |v(x,t)|^2 e^{2s\varphi(x,t)} dx$$

for $s \geq s_0$ and $0 \leq t \leq T$. Integrating this inequality in t over $(0, T)$, we have

$$\int_Q \left(\frac{1}{s} |\nabla p|^2 + s|p|^2 \right) e^{2s\varphi} dxdt$$

$$\leq C \int_Q (|F|^2 + |\partial_t h|^2 + |\nabla h|^2 + |h|^2) e^{2s\varphi} dxdt + C \int_Q |v|^2 e^{2s\varphi} dxdt \tag{2.7}$$

for all $s \geq s_0$.

Next, regarding $F - \nabla p$ in (2.2) as non-homogeneous term, we apply a Carleman estimate for the parabolic operator $\partial_t v - \kappa \Delta v + (A \cdot \nabla) v$ (e.g., Theorem 3.1 in Yamamoto [21]) to (2.2):

$$\frac{1}{s} \int_Q \left\{ \frac{1}{s} \left(|\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i \partial_j v|^2 \right) + s|\nabla v|^2 + s^3 |v|^2 \right\} e^{2s\varphi} dxdt$$

$$\leq C \int_Q \frac{1}{s} |\nabla p|^2 e^{2s\varphi} dxdt + C \int_Q \frac{1}{s} |F|^2 e^{2s\varphi} dxdt. \tag{2.8}$$

Substituting (2.7) into (2.8), we obtain

$$\int_Q \left\{ \frac{1}{s^2} \left(|\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i \partial_j v|^2 \right) + |\nabla v|^2 + s^2 |v|^2 \right\} e^{2s\varphi} dxdt$$

$$\leq C \int_Q |F|^2 e^{2s\varphi} dxdt + C \int_Q (|\partial_t h|^2 + |\nabla h|^2 + |h|^2) e^{2s\varphi} dxdt$$

$$+ C \int_Q |v|^2 e^{2s\varphi} dxdt + \frac{C}{s} \int_Q |F|^2 e^{2s\varphi} dxdt.$$

Choosing $s_0 > 0$ large, we can absorb the third term on the right-hand side into the left-hand side, again with (2.7), we complete the proof of Lemma 2. ■

Second Step.

Without loss of generality, we can assume that $d > 0$ in Ω because we replace d by $d + C_0$ with large constant $C_0 > 0$ if necessary.

In this step, we will prove

Lemma 3.

There exist constants $s_0 > 0$ and $C > 0$ such that

$$\|(v, p)\|^2_{\mathcal{X}_s(D)}$$

$$\leq C\int_D |F|^2 e^{2s\varphi}dxdt + C\int_D (|h|^2+|\nabla_{x,t}h|^2)e^{2s\varphi}dxdt$$
$$+Ce^{Cs}(\|v\|^2_{L^2(0,T;H^{\frac{3}{2}}(\Gamma))} + \|\partial_t v\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} + \|\partial_\nu v\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} + \|p\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))})$$

for all $s \geq s_0$ and $(v,p) \in H^{2,1}(D)\times H^{1,0}(D)$ satisfying (1.1), (1.6) and

$$div\, v = h \quad in\ D. \tag{2.9}$$

Proof of Lemma 3. We take the zero extensions of v, p, A, F to Q from D and by the same letters we denote them:

$$v = \begin{cases} v & \text{on } \overline{D}, \\ 0 & \text{in } Q\setminus D, \end{cases} \qquad p = \begin{cases} p & \text{on } \overline{D}, \\ 0 & \text{in } Q\setminus D, \text{ etc.} \end{cases}$$

By (1.6) we easily see that

$$\partial_i v = \begin{cases} \partial_i v & \text{on } \overline{D}, \\ 0 & \text{in } Q\setminus D, \end{cases} \qquad \partial_t v = \begin{cases} \partial_t v & \text{on } \overline{D}, \\ 0 & \text{in } Q\setminus D, \end{cases} \qquad \partial_i\partial_j v = \begin{cases} \partial_i\partial_j v, & \text{on } \overline{D}, \\ 0, & \text{in } Q\setminus D, \end{cases}$$

and

$$\partial_i p = \begin{cases} \partial_i p, & \text{on } \overline{D}, \\ 0, & \text{in } Q\setminus D \end{cases}$$

for $1 \leq i,j \leq n$. Moreover, since $v = 0$ on $\partial D \setminus (\Gamma\times(0,T))$ by (1.6), setting $h = \begin{cases} h & \text{on } \overline{D}, \\ 0 & \text{in } Q\setminus D \end{cases}$, we see that $h \in H^{1,1}(Q)$ and

$$\operatorname{div} v = h \quad \text{in } Q \tag{2.10}$$

and

$$\partial_t v = \kappa\Delta v + (A\cdot\nabla)v + \nabla p + F \quad \text{in } Q. \tag{2.11}$$

By the Sobolev extension theorem, there exist $\widetilde{p} \in L^2(0,T;H^1(\Omega))$ and $v \in H^{2,1}(Q)$ such that

$$\begin{cases} \widetilde{v} = v,\ \partial_\nu\widetilde{v} = \partial_\nu v,\ \widetilde{p} = p \quad \text{on } \partial\Omega\times(0,T), \\ \text{supp}\ \widetilde{v}(x,\cdot) \subset (0,T) \text{ for almost all } x\in\Omega \end{cases} \tag{2.12}$$

and

$$\|\widetilde{v}\|_{H^{2,1}(Q)} + \|\partial_t\widetilde{v}\|_{L^2(0,T;H^1(\Omega))} + \|\widetilde{p}\|_{L^2(0,T;H^1(\Omega))}$$

$$\leq C(\|v\|_{L^2(0,T;H^{\frac{3}{2}}(\Gamma))} + \|\partial_t v\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} + \|\partial_\nu v\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} + \|p\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))}). \quad (2.13)$$

The last condition in (2.12) can be seen by $v(\cdot,0) = v(\cdot,T) = 0$ in Ω which follows from (1.6).

We set

$$u = v - \widetilde{v}, \quad q = p - \widetilde{p} \quad \text{in } Q.$$

Then, in view of (2.10) - (2.12), we have

$$|u| = |\nabla u| = |q| = 0 \quad \text{on } \partial\Omega \times (0,T) \quad (2.14)$$

and

$$\partial_t u - \kappa\Delta u + \nabla q + (A\cdot\nabla)u = F - (\partial_t\widetilde{v} - \kappa\Delta\widetilde{v} + (A\cdot\nabla)\widetilde{v} + \nabla\widetilde{p}) =: G \quad \text{in } Q, \quad (2.15)$$

$$\operatorname{div} u = h - \operatorname{div}\widetilde{v} \in H^{1,1}(Q). \quad (2.16)$$

We choose a bounded domain $\widetilde{\Omega}$ with smooth boundary $\partial\widetilde{\Omega}$ such that $\widetilde{\Omega} \supset \Omega$, $\overline{\Gamma} = \partial\Omega \cap \widetilde{\Omega}$ and $\partial\widetilde{\Omega} \cap \overline{\Omega} = \partial\Omega \setminus \Gamma$. In other words, the domain $\widetilde{\Omega}$ is constructed by expanding Ω only over Γ to the exterior such that the boundary $\partial\widetilde{\Omega}$ is smooth. We set

$$\widetilde{Q} = \widetilde{\Omega} \times (0,T).$$

Let us recall that d satisfies (1.4). Since we can further choose $\widetilde{\Omega}$ such that $\widetilde{\Omega}\setminus\Omega$ is included in a sufficiently small ball, we see that there exists an extension $\widetilde{d}$ in $\widetilde{\Omega}$ of d satisfying $|\nabla\widetilde{d}| > 0$ in $\widetilde{\Omega}$.

We take the zero extensions of u, q, A, G and $h - \operatorname{div}\widetilde{v}$ to $\widetilde{\Omega}$ and by the same letters we denote them. Therefore by (2.14) - (2.16), the zero extensions of u and $h - \operatorname{div}\widetilde{v}$ satisfies

$$\operatorname{div} u = h - \operatorname{div}\widetilde{v} \in H^{1,1}(\widetilde{Q}) \quad (2.17)$$

and

$$\partial_t u - \kappa\Delta u + \nabla q + (A\cdot\nabla)u = G \quad \text{in } \widetilde{Q}. \quad (2.18)$$

By the zero extensions and (1.6), we obtain

$$u(\cdot,0) = u(\cdot,T) = 0 \quad \text{in } \widetilde{\Omega},$$

$$|u| = |\nabla u| = |q| = 0 \quad \text{on } \partial\widetilde{\Omega} \times (0,T). \quad (2.19)$$

Therefore, by noting (2.19), we apply Lemma 2 to (2.17) and (2.18), and we obtain

$$\|(u,q)\|^2_{\mathcal{X}_s(\tilde{Q})} \leq C\int_{\widetilde{Q}} |G|^2 e^{2s\varphi}\, dxdt$$

$$+C\int_{\widetilde{Q}}(|h-\operatorname{div}\widetilde{v}|^2+|\nabla_{x,t}(h-\operatorname{div}\widetilde{v})|^2)e^{2s\varphi}dxdt$$

for $s \ge s_0$. Hence

$$\begin{aligned}
&\|(v-\tilde{v},p-\tilde{p})\|^2_{\mathcal{X}_s(Q)}\\
\le& C\int_Q |F|^2 e^{2s\varphi}dxdt\\
+&C\int_Q |\partial_t\widetilde{v}-\kappa\Delta\widetilde{v}+(A\cdot\nabla)\widetilde{v}+\nabla\widetilde{p}|^2e^{2s\varphi}dxdt\\
+&C\int_Q\left(|h|^2+|\nabla_{x,t}h|^2+|\nabla\widetilde{v}|^2+\sum_{i,j=1}^n|\partial_i\partial_j\widetilde{v}|^2+|\nabla(\partial_t\widetilde{v})|^2\right)e^{2s\varphi}dxdt.
\end{aligned}$$

Using $|\partial_t v|^2 \le 2|\partial_t\widetilde{v}|^2 + 2|\partial_t(v-\widetilde{v})|^2$, etc. on the left-hand side, we have

$$\begin{aligned}
&\|(v,p)\|^2_{\mathcal{X}_s(Q)} \le 2\|(\tilde{v},\tilde{p})\|^2_{\mathcal{X}_s(Q)}\\
+&2C\int_Q |F|^2 e^{2s\varphi}dxdt\\
+&2C\int_Q |\partial_t\widetilde{v}-\kappa\Delta\widetilde{v}+(A\cdot\nabla)\widetilde{v}+\nabla\widetilde{p}|^2e^{2s\varphi}dxdt\\
+&2C\int_Q\left(|h|^2+|\nabla_{x,t}h|^2+|\nabla\widetilde{v}|^2+\sum_{i,j=1}^n|\partial_i\partial_j\widetilde{v}|^2+|\nabla(\partial_t\widetilde{v})|^2\right)e^{2s\varphi}dxdt\\
\le& C\int_Q |F|^2 e^{2s\varphi}dxdt\\
+&Ce^{Cs}(\|\widetilde{v}\|^2_{H^{2,1}(Q)}+\|\widetilde{p}\|^2_{H^{1,0}(Q)}+\|\nabla\partial_t\widetilde{v}\|^2_{L^2(Q)})\\
+&C\int_Q(|h|^2+|\nabla_{x,t}h|^2)e^{2s\varphi}dxdt
\end{aligned}$$

for $s \ge s_0$. Since F and h are zero outside of D, in view of (2.13), the proof of Lemma 3 is completed. ■

Third Step.

For $r > 0$ and $x_0 \in \mathbb{R}^n$, we set $B_r(x_0) := \{x \in \mathbb{R}^n;\ |x-x_0| < r\}$. Then we prove

Lemma 4.

Let $v \in H^2(\Omega)$ and $p \in H^1(\Omega)$.

(1) Case $n = 3$*: For any $x_0 \in \partial\Omega$, there exist $r > 0$ and a 10×10 matrix $A \in C^1(\overline{B_r(x_0)})$ such that*

$$\partial\Omega \cap B_r(x_0) = \{x(\theta_1,\theta_2);\ (\theta_1,\theta_2) \in D_1\}$$

where $x(\theta_1,\theta_2) = (x_1(\theta_1,\theta_2), x_2(\theta_1,\theta_2), x_3(\theta_1,\theta_2)) \in \mathbb{R}^3$*,* $D_1 \subset \mathbb{R}^2$ *is a bounded domain and the functions* x_1, x_2, x_3 *with respect to* θ_1, θ_2 *are in* $C^2(\overline{D_1})$ *and*

$$det\, A(x(\theta_1,\theta_2)) \ne 0, \quad (\theta_1,\theta_2) \in \overline{D_1}$$

and

$$A(x(\theta_1,\theta_2)) \begin{pmatrix} (\nabla_x v_1)(x(\theta_1,\theta_2)) \\ (\nabla_x v_2)(x(\theta_1,\theta_2)) \\ (\nabla_x v_3)(x(\theta_1,\theta_2)) \\ p(x(\theta_1,\theta_2)) \end{pmatrix} = \begin{pmatrix} \nabla_{\theta_1,\theta_2}(v_1(x(\theta_1,\theta_2))) \\ \nabla_{\theta_1,\theta_2}(v_2(x(\theta_1,\theta_2))) \\ \nabla_{\theta_1,\theta_2}(v_3(x(\theta_1,\theta_2))) \\ (\sigma(v,p)\nu)(x(\theta_1,\theta_2)) \\ (div\, v)(x(\theta_1,\theta_2)) \end{pmatrix}, \quad (\theta_1,\theta_2) \in D_1.$$

(2) Case $n = 2$*: For any* $x_0 \in \partial\Omega$*, there exist* $r > 0$ *and a* 5×5 *matrix* $A \in C^1(\overline{B_r(x_0)})$ *such that*

$$\partial\Omega \cap B_r(x_0) = \{x(\theta_1);\ \theta_1 \in I_1\}$$

where $x(\theta_1) := (x_1(\theta_1), x_2(\theta_1)) \in \mathbb{R}^2$*,* $I_1 \subset \mathbb{R}$ *is an open interval, and the functions* x_1, x_2 *are in* $C^2(\overline{I_1})$*, and*

$$det\, A(x(\theta_1)) \ne 0, \quad \theta_1 \in \overline{I_1}$$

and

$$A(x(\theta_1)) \begin{pmatrix} (\nabla_x v_1)(x(\theta_1)) \\ (\nabla_x v_2)(x(\theta_1)) \\ p(x(\theta_1)) \end{pmatrix} = \begin{pmatrix} \frac{d}{d\theta_1} v_1(x(\theta_1)) \\ \frac{d}{d\theta_1} v_2(x(\theta_1)) \\ (\sigma(v,p)\nu)(x(\theta_1)) \\ (div\, v)(x(\theta_1)) \end{pmatrix}, \quad \theta_1 \in I_1.$$

Remark. The lemma guarantees that the boundary data $(v, \partial_\nu v, p)$ and $(v, \sigma(v,p)\nu)$ are equivalent (e.g., Imanuvilov and Yamamoto [12]). As related papers on inverse boundary value problems for the Navier-Stokes equations in view of this equivalence, see Imanuvilov and Yamamoto [11], Lai, Uhlmann and Wang [15].

Proof of Lemma 4. We prove only in the case of $n = 3$. The case of $n = 2$ is similar and simpler. It is sufficient to consider only on a sufficiently small subboundary Γ_0 of $\partial\Omega$. Without loss of generality, we can assume that Γ_0 is represented by by $(x_1, x_2, \gamma(x_1,x_2))$ where $\gamma \in C^2(\overline{D_1})$, $\theta_1 = x_1$, $\theta_2 = x_2$, $x_3 = \gamma(x_1,x_2)$ for $(x_1,x_2) \in D_1$. Moreover we assume that Ω is located upper $x_3 = \gamma(x_1,x_2)$.

By the density argument, we can assume that $v \in C^1(\overline{\Omega})$ and $p \in C(\overline{\Omega})$.

We set $\gamma_1; = \partial_1\gamma$ and $\gamma_2 := \partial_2\gamma$. On Γ_0, we have

$$\nu(x) = \frac{1}{1+\gamma_1^2+\gamma_2^2}\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ -1 \end{pmatrix}. \tag{2.20}$$

By the definition, we have

$$\sigma(v,p)\nu = \kappa \begin{pmatrix} 2\partial_1 v_1 - \frac{p}{\kappa} & \partial_1 v_2 + \partial_2 v_1 & \partial_1 v_3 + \partial_3 v_1 \\ \partial_1 v_2 + \partial_2 v_1 & 2\partial_2 v_2 - \frac{p}{\kappa} & \partial_2 v_3 + \partial_3 v_2 \\ \partial_1 v_3 + \partial_3 v_1 & \partial_2 v_3 + \partial_3 v_2 & 2\partial_3 v_3 - \frac{p}{\kappa} \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{2.21}$$

$$=: \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}.$$

We further set

$$q_4 := (\operatorname{div} v)(x_1, x_2, \gamma(x_1,x_2)),$$

$$g_k(x_1,x_2) := v_k(x_1,x_2,\gamma(x_1,x_2)), \quad k = 1,2,3.$$

Then

$$\partial_1 g_k = \partial_1 v_k + \gamma_1(\partial_3 v_k)(x_1,x_2,\gamma(x_1,x_2)),$$

$$\partial_2 g_k = \partial_2 v_k + \gamma_2(\partial_3 v_k)(x_1,x_2,\gamma(x_1,x_2)),$$

that is,

$$\begin{cases} \partial_1 v_k(x_1,x_2,\gamma(x_1,x_2)) = \partial_1 g_k - \gamma_1(\partial_3 v_k)(x_1,x_2,\gamma(x_1,x_2)), \\ \partial_2 v_k(x_1,x_2,\gamma(x_1,x_2)) = \partial_2 g_k - \gamma_2(\partial_3 v_k)(x_1,x_2,\gamma(x_1,x_2)), \quad k = 1,2,3, \end{cases} \tag{2.22}$$

and

$$(\partial_3 v_3)(x_1,x_2,\gamma(x_1,x_2)) = q_4 - (\partial_1 v_1 + \partial_2 v_2)(x_1,x_2,\gamma(x_1,x_2)) \tag{2.23}$$

for $(x_1,x_2) \in D_1$. Setting

$$\begin{cases} h_1(x_1,x_2) = (\partial_3 v_1)(x_1,x_2,\gamma(x_1,x_2)), \\ h_2(x_1,x_2) = (\partial_3 v_2)(x_1,x_2,\gamma(x_1,x_2)), \end{cases} \tag{2.24}$$

by (2.22) and (2.23) we obtain

$$(\partial_3 v_3)(x_1,x_2,\gamma(x_1,x_2)) = q_4 - (\partial_1 v_1 + \partial_2 v_2)(x_1,x_2,\gamma(x_1,x_2))$$

$$
\begin{aligned}
&=(q_4-\partial_1 g_1-\partial_2 g_2)(x_1,x_2)+(\gamma_1\partial_3 v_1+\gamma_2\partial_3 v_2)(x_1,x_2,\gamma(x_1,x_2))\\
&=: g_0(x_1,x_2)+(\gamma_1\partial_3 v_1+\gamma_2\partial_3 v_2)(x_1,x_2,\gamma(x_1,x_2))\\
&= (g_0+\gamma_1 h_1+\gamma_2 h_2)(x_1,x_2)
\end{aligned}
\tag{2.25}
$$

and so

$$
\begin{cases}
(\partial_1 v_1)(x_1,x_2,\gamma(x_1,x_2)) = (\partial_1 g_1-\gamma_1 h_1)(x_1,x_2,\gamma(x_1,x_2)),\\
(\partial_2 v_1)(x_1,x_2,\gamma(x_1,x_2)) = (\partial_2 g_1-\gamma_2 h_1)(x_1,x_2,\gamma(x_1,x_2)),\\
(\partial_1 v_2)(x_1,x_2,\gamma(x_1,x_2)) = (\partial_1 g_2-\gamma_1 h_2)(x_1,x_2,\gamma(x_1,x_2)),\\
(\partial_2 v_2)(x_1,x_2,\gamma(x_1,x_2)) = (\partial_2 g_2-\gamma_2 h_2)(x_1,x_2,\gamma(x_1,x_2)),\\
(\partial_1 v_3)(x_1,x_2,\gamma(x_1,x_2)) = \partial_1 g_3-\gamma_1 g_0-\gamma_1^2 h_1-\gamma_1\gamma_2 h_2,\\
(\partial_2 v_3)(x_1,x_2,\gamma(x_1,x_2)) = \partial_2 g_3-\gamma_2 g_0-\gamma_1\gamma_2 h_1-\gamma_2^2 h_2, \quad (x_1,x_2)\in D_1.
\end{cases}
\tag{2.26}
$$

On the other hand, (2.21) yields

$$
\frac{1+\gamma_1^2+\gamma_2^2}{\kappa}q_1 = (2\gamma_1\partial_1 v_1+\gamma_2\partial_1 v_2+\gamma_2\partial_2 v_1-\partial_1 v_3-\partial_3 v_1)(x_1,x_2,\gamma(x_1,x_2))-\frac{\gamma_1}{\kappa}p,
$$

$$
\frac{1+\gamma_1^2+\gamma_2^2}{\kappa}q_2 = (\gamma_1\partial_1 v_2+\gamma_2\partial_2 v_1+2\gamma_2\partial_2 v_2-\partial_2 v_3-\partial_3 v_2)(x_1,x_2,\gamma(x_1,x_2))-\frac{\gamma_2}{\kappa}p
$$

and

$$
\frac{1+\gamma_1^2+\gamma_2^2}{\kappa}q_3 = (\gamma_1\partial_1 v_3+\gamma_1\partial_3 v_1+\gamma_2\partial_2 v_3+\gamma_2\partial_3 v_2-2\partial_3 v_3)(x_1,x_2,\gamma(x_1,x_2))+\frac{1}{\kappa}p, \quad (x_1,x_2)\in D_1.
$$

Substitute (2.25) and (2.26), we have

$$
\begin{cases}
\frac{1+\gamma_1^2+\gamma_2^2}{\kappa}q_1 = -(1+\gamma_1^2+\gamma_2^2)h_1-\frac{\gamma_1}{\kappa}p+G_1,\\
\frac{1+\gamma_1^2+\gamma_2^2}{\kappa}q_2 = -(1+\gamma_1^2+\gamma_2^2)h_2-\frac{\gamma_2}{\kappa}p+G_2,\\
\frac{1+\gamma_1^2+\gamma_2^2}{\kappa}q_3 = -\gamma_1(1+\gamma_1^2+\gamma_2^2)h_1-\gamma_2(1+\gamma_1^2+\gamma_2^2)h_2+\frac{1}{\kappa}p+G_3.
\end{cases}
\tag{2.27}
$$

Here $G_k,\ k=1,2,3,$ are linear combinations of $\partial_j g_k, q_1,q_2,q_3,q_4,\ j=1,2, k=1,2,3,$ with coefficients given by γ and its first-order derivatives. We can uniquely solve (2.27) with respect to h_1,h_2,p:

$$
\begin{pmatrix}
h_1(x_1,x_2)\\
h_2(x_2,x_2)\\
p(x_1,x_2,\gamma(x_1,x_2))
\end{pmatrix}
= \widetilde{A}(x_1,x_2)
\begin{pmatrix}
\frac{1+\gamma_1^2+\gamma_2^2}{\kappa}q_1-G_1\\
\frac{1+\gamma_1^2+\gamma_2^2}{\kappa}q_2-G_2\\
\frac{1+\gamma_1^2+\gamma_2^2}{\kappa}q_3-G_3
\end{pmatrix}, \quad (x_1,x_2)\in D_1. \tag{2.28}
$$

Here $\widetilde{A} \in C^1(\overline{D_1})$ and $\det \widetilde{A} \neq 0$ on $\overline{D_1}$. The equations (2.25), (2.26) and (2.28) imply the existence of a 10×10 matrix $A \in C^1(\overline{D_1})$ satisfying the conditions in the lemma. Thus the proof of Lemma 4 is completed. ■

Now, in terms of Lemmata 3 and 4, we complete the proof of Theorem 1 as follows. We consider only the case of $n = 3$. Without loss of generality, Γ is given by $\Gamma = \{(x_1, x_2, \gamma(x_1, x_2));\ x_1, x_2 \in D_1\}$ with $\gamma \in C^2(\overline{D_1})$.
We set $\nabla_{x_1,x_2} v = (\partial_1 v_1, \partial_2 v_1, \partial_1 v_2, \partial_2 v_2, \partial_1 v_3, \partial_2 v_3)^T$. Then, by Lemmata 2 and 3, we have

$$\begin{pmatrix} \partial_\nu v(x_1, x_2, \gamma(x_1, x_2)) \\ p(x_1, x_2, \gamma(x_1, x_2)) \end{pmatrix} = \begin{pmatrix} \frac{1}{1+\gamma_1^2+\gamma_2^2}((\partial_1\gamma)\partial_1 v + (\partial_2\gamma)\partial_2 v - \partial_3 v)(x_1, x_2, \gamma(x_1, x_2)) \\ p(x_1, x_2, \gamma(x_1, x_2)) \end{pmatrix}$$

$$= B_1(x_1, x_2) \begin{pmatrix} (\nabla_{x_1,x_2} v)(x_1, x_2, \gamma(x_1, x_2)) \\ (\sigma(v, p)\nu)(x_1, x_2, \gamma(x_1, x_2)) \end{pmatrix}, \quad (x_1, x_2) \in D_1,$$

with a 4×6 matrix $B_1 \in C^1(\overline{D_1})$. Therefore

$$\|\partial_\nu v(\cdot, t)\|_{H^1(\Gamma)} + \|p(\cdot, t)\|_{H^1(\Gamma)} = \left\| B \begin{pmatrix} \nabla_{x_1,x_2} v \\ \sigma(v, p)\nu \end{pmatrix} (\cdot, t) \right\|_{H^1(\Gamma)} \le C \left\| \begin{pmatrix} \nabla_{x_1,x_2} v \\ \sigma(v, p)\nu \end{pmatrix} (\cdot, t) \right\|_{H^1(\Gamma)}$$

and

$$\|\partial_\nu v(\cdot, t)\|_{L^2(\Gamma)} \le C \left\| \begin{pmatrix} \nabla_{x_1,x_2} v \\ \sigma(v, p)\nu \end{pmatrix} (\cdot, t) \right\|_{L^2(\Gamma)}$$

by $B \in C^1(\overline{D_1})$. Consequently the interpolation inequality (e.g., Theorem 7.7 (p.36) in Lions and Magenes [17]) yields

$$\|\partial_\nu v(\cdot, t)\|_{H^{\frac{1}{2}}(\Gamma)} + \|p(\cdot, t)\|_{H^{\frac{1}{2}}(\Gamma)} \le \left\| \begin{pmatrix} \nabla_{x_1,x_2} v \\ \sigma(v, p)\nu \end{pmatrix} (\cdot, t) \right\|_{H^{\frac{1}{2}}(\Gamma)}$$

for $0 \le t \le T$. Hence

$$\|\partial_\nu v\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} + \|p(\cdot, t)\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} \le C(\|v\|_{L^2(0,T;H^1(\Gamma))} + \|\sigma(v, p)\nu\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))}).$$

With this, Lemma 3 completes the proof of Theorem 1. ■

3 Conditional stability for the lateral Cauchy problem

In this section, we discuss

lateral Cauchy problem

We are given a suboundary Γ of $\partial\Omega$ arbitrarily. Let $(v,p) \in H^{2,1}(Q) \times H^{1,0}(Q)$ satisfy (1.1) and (1.2). Determine (v,p) in some subdomain of Q by $(v, \sigma(v,p)\nu)$ on $\Gamma \times (0,T)$.

In the case of the parabolic equation, there are very many works, and here we do not list up comprehensively and as restricted references, see Landis [16], Mizohata [18], Saut and Scheurer [19], Sogge [20]. See also the monographs Beilina and Klibanov [1], Isakov [13], Klibanov and Timonov [14].

Combining a Carleman estimate and a cut-off function, we can prove

Proposition 1.

Let $\varphi(x,t)$ be given in Theorem 1. We set

$$Q(\varepsilon) = \{(x,t) \in \Omega \times (0,T); \varphi(x,t) > \varepsilon\}$$

with $\varepsilon > 0$. Moreover we assume that

$$\overline{Q(0)} \subset Q \cup (\Gamma \times [0,T])$$

with subboundary $\Gamma \subset \partial\Omega$. Then for any small $\varepsilon > 0$, there exist constants $C > 0$ and $\theta \in (0,1)$ such that

$$\|v\|_{H^{2,1}(Q(\varepsilon))} + \|p\|_{H^{1,0}(Q(\varepsilon))} \le C(\|v\|^{1-\theta}_{H^{1,1}(Q)} + \|p\|_{L^2(Q)})G^\theta + CG,$$

where we set

$$G^2 := \|F\|^2_{L^2(Q)} + \|v\|^2_{L^2(0,T;H^{\frac{3}{2}}(\Gamma))} + \|\partial_t v\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} + \|\sigma(v,p)\nu\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))}.$$

As for the proof of Proposition 1, see Theorem 3.2.2 in section 3.2 of [13] for example.

Proposition 1 gives an estimate of the solution in $Q(\varepsilon)$ by data on $\Gamma \times (0,T)$, and $Q(\varepsilon)$ and Γ are determined by an a priori given function $d(x)$. Therefore the proposition does not give a suitable answer to our lateral Cauchy problem as stated above, where we are requested to estimate the solution by data on as a small subboundary $\Gamma \times (0,T)$ as possible.

In fact, in this section, we prove

Theorem 2 (conditional stability)

Let $\Gamma \subset \partial\Omega$ be an arbitrary non-empty subboundary of $\partial\Omega$. For any $\varepsilon > 0$ and an arbitrary bounded domain Ω_0 such that $\overline{\Omega_0} \subset \Omega \cup \Gamma$, $\partial\Omega_0 \cap \partial\Omega$ is a non-empty open subset of $\partial\Omega$ and $\partial\Omega_0 \cap \partial\Omega \subsetneqq \Gamma$, there exist constants $C > 0$ and $\theta \in (0,1)$ such that

$$\|v\|_{H^{2,1}(\Omega_0\times(\varepsilon,T-\varepsilon))} + \|p\|_{H^{1,0}(\Omega_0\times(\varepsilon,T-\varepsilon))}$$

$$\le C(\|v\|_{H^{1,1}(Q)}+\|p\|_{L^2(Q)})^{1-\theta}(\|F\|_{L^2(Q)}+\|v\|_{L^2(0,T;H^{\frac{3}{2}}(\Gamma))}+\|v\|_{H^1(0,T;H^{\frac{1}{2}}(\Gamma))}+\|\sigma(v,p)\nu\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))})^{\theta}$$
$$+C(\|F\|_{L^2(Q)} + \|v\|_{L^2(0,T;H^{\frac{3}{2}}(\Gamma))} + \|\partial_t v\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} + \|\sigma(v,p)\nu\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))}). \quad (3.1)$$

In Theorem 2, in order to estimate (v,p), we have to assume a priori bounds of $\|v\|_{H^{1,1}(Q)}$ and $\|p\|_{L^2(Q)}$. Thus estimate (3.1) is called a conditional stability estimate. We note that (3.1) is rewritten as

$$\|v\|_{H^{2,1}(\Omega_0\times(\varepsilon,T-\varepsilon))} + \|p\|_{H^{1,0}(\Omega_0\times(\varepsilon,T-\varepsilon))}$$
$$=O((\|F\|_{L^2(Q)} + \|v\|_{L^2(0,T;H^{\frac{3}{2}}(\Gamma))} + \|v\|_{H^1(0,T;H^{\frac{1}{2}}(\Gamma))} + \|\sigma(v,p)\nu\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))})^{\theta})$$

as $\|F\|_{L^2(Q)} + \|v\|_{L^2(0,T;H^{\frac{3}{2}}(\Gamma))} + \|v\|_{H^1(0,T;H^{\frac{1}{2}}(\Gamma))} + \|\sigma(v,p)\nu\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} \longrightarrow 0$. Thus the estimate indicates stability of Hölder type.

For the homogeneous Stokes equations:

$$\partial_t v - \Delta v + \nabla p = 0, \quad \operatorname{div} v = 0 \quad \text{in } Q,$$

Boulakia [2] (Proposition 2) proved the conditional stability in $\Omega_0 \times (\varepsilon, T-\varepsilon)$ on the basis of a Carleman estimate in [2]. The norm of boundary data in [2] is stronger than our chosen norm.

The theorem does not directly give an estimate when $\Omega_0 = \Omega$, but we can derive an estimate in Ω by an argument similar to Theorem 5.2 in Yamamoto [21] and we do not discuss details. Boulakia [2] (Theorem 1) established a conditional stability estimate up to $\partial\Omega$ by boundary or interior data. The argument is based on the interior estimate in $\Omega_0 \times (\varepsilon, T-\varepsilon)$ and an argument similar to Theorem 5.2 in [21].

Theorem 2 immediately implies the global uniqueness of the solution:

Corollary.

Let $\Gamma \subset \partial\Omega$ be an arbitrarily fixed subboundary. If $(v,p) \in H^{2,1}(Q) \times H^{1,0}(Q)$ satisfies

(1.1) and (1.2), and $v = \sigma(v,p)\nu = 0$ on $\Gamma\times(0,T)$, then $|v| = \sigma(v,p)\nu = 0$ in $\Omega\times(0,T)$.

Proof of Theorem 2. Once a relevant Carleman estimate for the Navier-Stokes equations is proved, the proof is similar to Theorem 5.1 in [21]. Thus, according to Ω_0 and Γ, we have to choose a suitable weight function φ. For this, we show

Lemma 5.

Let ω be an arbitrarily fixed subdomain of Ω such that $\overline{\omega} \subset \Omega$. Then there exists a function $d \in C^2(\overline{\Omega})$ such that

$$d(x) > 0 \quad x \in \Omega, \quad d|_{\partial\Omega} = 0, \quad |\nabla d(x)| > 0, \quad x \in \overline{\Omega \setminus \omega}.$$

For the proof, see Fursikov and Imanuvilov [7], Imanuvilov [8], Imanuvilov, Puel and Yamamoto [10].

We choose a bounded domain Ω_1 with smooth boundary such that

$$\Omega \subsetneqq \Omega_1, \quad \overline{\Gamma} = \overline{\partial\Omega \cap \Omega_1}, \quad \partial\Omega \setminus \Gamma \subset \partial\Omega_1, \tag{3.2}$$

and $\Omega_1 \setminus \overline{\Omega}$ contains some non-empty open set. We note that Ω_1 is constructed by taking a union of Ω and a domain $\widetilde{\Omega} \subset \mathbb{R}^n \setminus \overline{\Omega}$ such that $\widetilde{\Omega} \cap \partial\Omega = \Gamma$. Choosing $\overline{\omega} \subset \Omega_1 \setminus \overline{\Omega}$, and applying Lemma 5 to obtain $d \in C^2(\overline{\Omega_1})$ satisfying

$$d(x) > 0, \quad x \in \Omega_1, \quad d(x) = 0, \quad x \in \partial\Omega_1, \quad |\nabla d(x)| > 0, \quad x \in \overline{\Omega}. \tag{3.3}$$

Then, since $\overline{\Omega_0} \subset \Omega_1$, we can choose sufficiently large $N > 1$ such that

$$\{x \in \Omega_1;\, d(x) > \frac{4}{N}\|d\|_{C(\overline{\Omega_1})}\} \cap \overline{\Omega} \supset \Omega_0. \tag{3.4}$$

Moreover we choose sufficiently large $\beta > 0$ such that

$$\beta\varepsilon^2 < \|d\|_{C(\overline{\Omega_1})} < 2\beta\varepsilon^2. \tag{3.5}$$

We arbitrarily fix $t_0 \in [\sqrt{2}\varepsilon, T-\sqrt{2}\varepsilon]$. We set $\varphi(x,t) = e^{\lambda\psi(x,t)}$ with fixed large parameter $\lambda > 0$ and $\psi(x,t) = d(x) - \beta(t-t_0)^2$, $\mu_k = \exp\left(\lambda\left(\frac{k}{N}\|d\|_{C(\overline{\Omega_1})} - \frac{\beta\varepsilon^2}{N}\right)\right)$, $k = 1,2,3,4$, and $D = \{(x,t);\, x \in \overline{\Omega}, \quad \varphi(x,t) > \mu_1\}$.

Then we can verify that

$$\Omega_0 \times \left(t_0 - \frac{\varepsilon}{\sqrt{N}}, t_0 + \frac{\varepsilon}{\sqrt{N}}\right) \subset D \subset \overline{\Omega} \times (t_0 - \sqrt{2}\varepsilon, t_0 + \sqrt{2}\varepsilon). \tag{3.6}$$

In fact, let $(x,t) \in \Omega_0 \times \left(t_0 - \frac{\varepsilon}{\sqrt{N}}, t_0 + \frac{\varepsilon}{\sqrt{N}}\right)$. Then, by (3.4) we have $x \in \overline{\Omega}$ and $d(x) > \frac{4}{N}\|d\|_{C(\overline{\Omega_1})}$, so that

$$d(x) - \beta(t-t_0)^2 > \frac{4}{N}\|d\|_{C(\overline{\Omega_1})} - \frac{\beta\varepsilon^2}{N},$$

that is, $\varphi(x,t) > \mu_4$, which implies that $(x,t) \in D$ by the definition of D. Next let $(x,t) \in D$. Then $d(x) - \beta(t-t_0)^2 > \frac{1}{N}\|d\|_{C(\overline{\Omega_1})} - \frac{\beta\varepsilon^2}{N}$. Therefore

$$\|d\|_{C(\overline{\Omega_1})} - \frac{1}{N}\|d\|_{C(\overline{\Omega_1})} + \frac{\beta\varepsilon^2}{N} > \beta(t-t_0)^2.$$

Applying (3.5), we have $2\left(1-\frac{1}{N}\right)\beta\varepsilon^2 + \frac{\beta\varepsilon^2}{N} > \left(1-\frac{1}{N}\right)\|d\|_{C(\overline{\Omega_1})} + \frac{\beta\varepsilon^2}{N} > \beta(t-t_0)^2$, that is, $2\beta\varepsilon^2 > \beta(t-t_0)^2$, which implies that $t_0 - \sqrt{2}\varepsilon < t < t_0 + \sqrt{2}\varepsilon$. The verification of (3.6) is completed.

Next we have

$$\begin{cases} \partial D \subset \Sigma_1 \cup \Sigma_2, \\ \Sigma_1 \subset \Gamma \times (0,T), \quad \Sigma_2 = \{(x,t);\, x \in \Omega,\, \varphi(x,t) = \mu_1\}. \end{cases} \tag{3.7}$$

In fact, let $(x,t) \in \partial D$. Then $x \in \overline{\Omega}$ and $\varphi(x,t) \ge \mu_1$. We separately consider the cases $x \in \Omega$ and $x \in \partial\Omega$. First let $x \in \Omega$. If $\varphi(x,t) > \mu_1$, then (x,t) is an interior point of D, which is impossible. Therefore $\varphi(x,t) = \mu_1$, which implies $(x,t) \in \Sigma_2$. Next let $x \in \partial\Omega$. Let $x \in \partial\Omega \setminus \Gamma$. Then $x \in \partial\Omega_1$ by the third condition in (3.2), and $d(x) = 0$ by the second condition in (3.3). On the other hand, $\varphi(x,t) \ge \mu_1$ yields that

$$d(x) - \beta(t-t_0)^2 = -\beta(t-t_0)^2 \ge \frac{1}{N}\|d\|_{C(\overline{\Omega_1})} - \frac{\beta\varepsilon^2}{N},$$

that is, $0 \le \beta(t-t_0)^2 \le \frac{1}{N}(-\|d\|_{C(\overline{\Omega_1})} + \beta\varepsilon^2)$, which is impossible by (3.5). Therefore $x \in \Gamma$. By (3.6), we see that $0 < t < T$ and the verification of (3.7) is completed.

We apply Theorem 1 in D. Henceforth $C > 0$ denotes generic constants independent of s and choices of v, p. We need a cut-off function because we have no data on $\partial D \setminus (\Gamma \times (0,T))$. Let $\chi \in C^\infty(\mathbb{R}^{n+1})$ satisfying $0 \le \chi \le 1$ and

$$\chi(x,t) = \begin{cases} 1, & \varphi(x,t) > \mu_3, \\ 0, & \varphi(x,t) < \mu_2. \end{cases} \tag{3.8}$$

We set $y = \chi v$ and $q = \chi p$. Then, by (1.1) and (1.2), we have

$$\partial_t y - \kappa\Delta y + (A\cdot\nabla)y + (y\cdot\nabla)B + \nabla q$$

$$=\chi F + v\partial_t\chi - 2\kappa\nabla\chi\cdot\nabla v - \kappa(\Delta\chi)v + (A\cdot\nabla\chi)v + p(\nabla\chi) \quad \text{in } D$$

and

$$\operatorname{div} y = \nabla\chi\cdot v \quad \text{in } D.$$

By (3.7) and (3.8), we see that

$$|y| = |\nabla y| = |q| = 0 \qquad \text{on } \Sigma_2.$$

Hence Theorem 1 yields

$$\begin{aligned}
&\|(y,q)\|^2_{\mathcal{X}_s(D)} \le C\int_D |F|^2 e^{2s\varphi}dxdt \\
&+C\int_D |v\partial_t\chi - 2\kappa\nabla\chi\cdot\nabla v - \kappa(\Delta\chi)v + (A\cdot\nabla\chi)v + p(\nabla\chi)|^2 e^{2s\varphi}dxdt \\
&+C\int_D (|\nabla\chi\cdot v|^2 + |\nabla_{x,t}(\nabla\chi\cdot v)|^2)e^{2s\varphi}dxdt \\
&+Ce^{Cs}(\|\chi v\|^2_{L^2(0,T;H^{\frac{3}{2}}(\Gamma))} + \|\partial_t(\chi v)\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} + \|\sigma(\chi v,\chi p)\nu\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))})
\end{aligned} \tag{3.9}$$

for $s \ge s_0$. We can verify $\|\chi v\|_{H^\gamma(\Gamma)} \le C\|v\|_{H^\gamma(\Gamma)}$ with $\gamma = 0,1,2$, and for $j=\frac{1}{2}$ and $j=\frac{3}{2}$, the interpolation inequality yields

$$\|\chi v\|^2_{L^2(0,T;H^j(\Gamma))} \le C\|v\|^2_{L^2(0,T;H^j(\Gamma))}, \quad \|\partial_t(\chi v)\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} \le C\|\partial_t v\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))}.$$

Therefore, since

$$\sigma(\chi v,\chi p)\nu = \chi\sigma(v,p)\nu + \kappa((\partial_i\chi)v_j + (\partial_j\chi)v_i)_{1\le i,j\le n}\nu,$$

we have

$$\|\sigma(\chi v,\chi p)\nu\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} \le \|\sigma(v,p)\nu\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} + C\|v\|_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))}$$

by $\chi\in C^\infty(\mathbb{R}^{n+1})$. Hence

$$\begin{aligned}
&\|\chi v\|^2_{L^2(0,T;H^{\frac{3}{2}}(\Gamma))} + \|\partial_t(\chi v)\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} + \|\sigma(\chi v,\chi p)\nu\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} \\
\le& C(\|v\|^2_{L^2(0,T;H^{\frac{3}{2}}(\Gamma))} + \|\partial_t v\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} + \|\sigma(v,p)\nu\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))}).
\end{aligned}$$

We recall that

$$G^2 = \|F\|^2_{L^2(Q)} + \|v\|^2_{L^2(0,T;H^{\frac{3}{2}}(\Gamma))} + \|\partial_t v\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))} + \|\sigma(v,p)\nu\|^2_{L^2(0,T;H^{\frac{1}{2}}(\Gamma))}.$$

The integrands of the second and the third terms on the right-hand side of (3.9) do not vanish only if $\varphi(x,t) \le \mu_3$, because these coefficients include derivatives of χ as factors and by (3.8) vanish if $\varphi(x,t) > \mu_3$. Therefore

$$\begin{aligned}&|[\text{the second and the third terms on the right-hand side of (3.9)}]|\\ \le& C(\|v\|^2_{H^{1,1}(Q)} + \|p\|^2_{L^2(Q)})e^{2s\mu_3}.\end{aligned}$$

Consequently (3.9) yields

$$\|(y,q)\|^2_{\mathcal{X}_s(D)} \le C(\|v\|^2_{H^{1,1}(Q)} + \|p\|^2_{L^2(Q)})e^{2s\mu_3} + Ce^{Cs}G^2 \quad \forall s \ge s_0. \tag{3.10}$$

By (3.4) and the definition of D, we can directly verify that $(x,t) \in \Omega_0 \times \left(t_0 - \frac{\varepsilon}{\sqrt{N}}, t_0 + \frac{\varepsilon}{\sqrt{N}}\right)$ implies $\varphi(x,t) > \mu_4$. Therefore, noting (3.6) and (3.8), we see that

$$\begin{aligned}&\|(y,q)\|^2_{\mathcal{X}_s(D)} \ge \|(v,p)\|^2_{\mathcal{X}_s(\Omega_0\times(t_0-\frac{\varepsilon}{\sqrt{N}},t_0+\frac{\varepsilon}{\sqrt{N}}))}\\ \ge& e^{2s\mu_4}\int_{t_0-\frac{\varepsilon}{\sqrt{N}}}^{t_0+\frac{\varepsilon}{\sqrt{N}}}\int_{\Omega_0}\left\{\frac{1}{s^2}\left(|\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i\partial_j v|^2\right) + |\nabla v|^2 + s^2|v|^2 + \frac{1}{s}|\nabla p|^2 + s|p|^2\right\}dxdt.\end{aligned}$$

Hence (3.10) yields

$$\begin{aligned}&e^{2s\mu_4}\int_{t_0-\frac{\varepsilon}{\sqrt{N}}}^{t_0+\frac{\varepsilon}{\sqrt{N}}}\int_{\Omega_0}\left\{\frac{1}{s^2}\left(|\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i\partial_j v|^2\right) + |\nabla v|^2 + s^2|v|^2 + \frac{1}{s}|\nabla p|^2 + s|p|^2\right\}dxdt\\ \le& C(\|v\|^2_{H^{1,1}(Q)} + \|p\|^2_{L^2(Q)})e^{2s\mu_3} + Ce^{Cs}G^2.\end{aligned}$$

Therefore

$$\begin{aligned}&\int_{t_0-\frac{\varepsilon}{\sqrt{N}}}^{t_0+\frac{\varepsilon}{\sqrt{N}}}\int_{\Omega_0}\left\{\left(|\partial_t v|^2 + \sum_{i,j=1}^n |\partial_i\partial_j v|^2\right) + |\nabla v|^2 + |v|^2 + |\nabla p|^2 + |p|^2\right\}dxdt\\ \le& Cs^2e^{-2s(\mu_4-\mu_3)}(\|v\|^2_{H^{1,1}(Q)} + \|p\|^2_{L^2(Q)}) + Ce^{Cs}G^2 \quad \forall s \ge s_0.\end{aligned}$$

By $\sup_{s>0} se^{-s(\mu_4-\mu_3)} < \infty$, we estimate $se^{-2s(\mu_4-\mu_3)}$ by $e^{-s(\mu_4-\mu_3)}$ on the right-hand side. Moreover, replacing C by Ce^{Cs_0}, we can have

$$\begin{aligned}&\|v\|^2_{H^{2,1}\left(\Omega_0\times\left(t_0-\frac{\varepsilon}{\sqrt{N}},t_0+\frac{\varepsilon}{\sqrt{N}}\right)\right)} + \|p\|^2_{H^{1,0}\left(\Omega_0\times\left(t_0-\frac{\varepsilon}{\sqrt{N}},t_0+\frac{\varepsilon}{\sqrt{N}}\right)\right)}\\ &\qquad\le Ce^{-s(\mu_4-\mu_3)}(\|v\|^2_{H^{1,1}(Q)} + \|p\|^2_{L^2(Q)}) + Ce^{Cs}G^2\end{aligned} \tag{3.11}$$

for all $s \ge 0$. Let $m \in \mathbb{N}$ satisfy $\sqrt{2}\varepsilon + \frac{m\varepsilon}{\sqrt{N}} \le T - \sqrt{2}\varepsilon \le \sqrt{2}\varepsilon + \frac{(m+1)\varepsilon}{\sqrt{N}} \le T$.

We here notice that the constant C in (3.11) is independent also of t_0, provided that $\sqrt{2}\varepsilon \le t_0 \le T - \sqrt{2}\varepsilon$. In (3.11), taking $t_0 = \sqrt{2}\varepsilon + \frac{j\varepsilon}{\sqrt{N}}$, $j = 0, 1, 2, ..., m$ and summing up over j, we have

$$\|v\|^2_{H^{2,1}\left(\Omega_0\times\left(\sqrt{2}\varepsilon - \frac{\varepsilon}{\sqrt{N}}, T-\sqrt{2}\varepsilon-\frac{\varepsilon}{\sqrt{N}}\right)\right)} + \|p\|^2_{H^{1,0}\left(\Omega_0\times\left(\sqrt{2}\varepsilon - \frac{\varepsilon}{\sqrt{N}}, T-\sqrt{2}\varepsilon-\frac{\varepsilon}{\sqrt{N}}\right)\right)}$$
$$\le Ce^{-s(\mu_4-\mu_3)}(\|v\|^2_{H^{1,1}(Q)} + \|p\|^2_{L^2(Q)}) + Ce^{Cs}G^2$$

for all $s \ge 0$. Here we note that $T - \sqrt{2}\varepsilon \le \sqrt{2}\varepsilon + \frac{(m+1)\varepsilon}{\sqrt{N}}$ implies $T - \sqrt{2}\varepsilon - \frac{m\varepsilon}{\sqrt{N}} \le \sqrt{2}\varepsilon + \frac{1}{\sqrt{N}}\varepsilon$. Replacing $\left(\sqrt{2} + \frac{1}{\sqrt{N}}\right)\varepsilon$ by ε, we have

$$\|v\|^2_{H^{2,1}(\Omega_0\times(\varepsilon,T-\varepsilon))} + \|p\|^2_{H^{1,0}(\Omega_0\times(\varepsilon,T-\varepsilon))}$$
$$\le Ce^{-s(\mu_4-\mu_3)}(\|v\|^2_{H^{1,1}(Q)} + \|p\|^2_{L^2(Q)}) + Ce^{Cs}G^2 \tag{3.12}$$

for all $s \ge s_0$.

First let $G = 0$. Then letting $s \to \infty$ in (3.12), we see that $|v| = |p| = 0$ in $\Omega_0 \times (\varepsilon, T-\varepsilon)$, so that the conclusion of Theorem 2 holds true. Next let $G \ne 0$. First let $G \ge \|v\|_{H^{1,1}(Q)} + \|p\|_{L^2(Q)}$. Then (3.12) implies $\|v\|_{H^{2,1}(\Omega_0\times(\varepsilon,T-\varepsilon))} + \|p\|_{H^{1,0}(\Omega_0\times(\varepsilon,T-\varepsilon))} \le Ce^{Cs}G$ for $s \ge 0$, which already proves the theorem. Second let $G < \|v\|_{H^{1,1}(Q)} + \|p\|_{L^2(Q)}$. In order to make the right-hand side of (3.12) smaller, we choose $s > 0$ such that

$$e^{-s(\mu_4-\mu_3)}(\|v\|^2_{H^{1,1}(Q)} + \|p\|^2_{L^2(Q)}) = e^{Cs}G^2.$$

By $G \ne 0$, we can choose

$$s = \frac{1}{C+\mu_4-\mu_3}\log\frac{\|v\|^2_{H^{1,1}(Q)} + \|p\|^2_{L^2(Q)}}{G^2} > 0.$$

Then (3.12) gives

$$\|v\|^2_{H^{2,1}(\Omega_0\times(\varepsilon,T-\varepsilon))} + \|p\|^2_{H^{1,0}(\Omega_0\times(\varepsilon,T-\varepsilon))} \le 2C(\|v\|^2_{H^{1,1}(Q)} + \|p\|^2_{L^2(Q)})^{\frac{C}{C+\mu_4-\mu_3}}G^{\frac{2(\mu_4-\mu_3)}{C+\mu_4-\mu_3}}.$$

The the proof of Theorem 2 is completed. ■

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